

CLASSIFICATION OF FINITE DIMENSIONAL MODULES OF SINGLY ATYPICAL TYPE OVER THE LIE SUPERALGEBRAS $sl(m/n)$

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ABSTRACT. We classify the finite dimensional indecomposable $sl(m/n)$ -modules with at least a typical or singly atypical primitive weight. We do this classification not only for weight modules, but also for generalized weight modules. We obtain that such a generalized weight module is simply a module obtained by “linking” sub-quotient modules of generalized Kac-modules. By applying our results to $sl(m/1)$, we have in fact completely classified all finite dimensional $sl(m/1)$ -modules.

KEYWORDS: atypical, primitive, weight diagram, generalized Kac-module, chain.

I. INTRODUCTION

Because finite dimensional indecomposable modules over Lie superalgebras are not always simple, the representation theory of Lie superalgebras is more complicated than that of Lie algebras. Kac in Ref. 1 defined the induced modules $\overline{V}(\Lambda)$ for integral dominant weights Λ , which are referred to as Kac-modules by Van der Jeugt *et al* in Ref. 2. Kac divided the Kac-modules into two categories: *typical* or *atypical* according as they are simple or not, he also gave a necessary and sufficient condition for $\overline{V}(\Lambda)$ to be simple. Ref. 2 gave a character formula for singly atypical simple $sl(m/n)$ -modules. Hughes *et al* in Ref. 3 achieved much progress on the classification of composition factors of Kac-modules. However, the structure of a Kac-module is in general still an unsolved problem. More generally, the problem of classifying finite dimensional indecomposable modules, posted in Ref. 1, remains open.

We made a start in Ref. 4, by giving a complete classification of finite dimensional $sl(2/1)$ -modules. In this paper, we generalize the results to $sl(m/n)$. More precisely, we classify all finite dimensional $sl(m/n)$ -modules with at least a typical or singly atypical primitive weight. It may be worth mentioning that although our results here are similar to those in Ref. 4, the proofs are more interesting, more technical, and also more complicated since Lemma 2.6 in Ref. 4 which was crucial

¹Partly supported by a grant from Shanghai Jiaotong University

in the proof of that paper, is no longer valid for general $sl(m/n)$.

By introducing the weight diagram, we are able to obtain Theorem 2.9, a crucial preliminary result in our classification. Then in Sect. III, by classifying the weight diagram, we obtain our main result of this section in Theorem 3.8, so that we have a clear picture of a module. In Sect. IV, by looking deep into generalized weight modules, we can understand these modules better. Then by a strict and complete proof, we find out in Theorem 4.9 that such a module is nothing but a module obtained by “linking” some sub-quotient modules of generalized Kac-modules.

By applying our results to $sl(m/1)$, we have efficiently classified all finite dimensional $sl(m/1)$ -modules.

We would like to point out that it may be possible to use our method to classify general indecomposable modules as long as we have a better understanding of the structure of a Kac-module in general.

II. THE LIE SUPERALGEBRA $sl(m/n)$ AND PRELIMINARY RESULTS

Let G denote the space $sl(m+1/n+1)$ consisting of $(m+n+2) \times (m+n+2)$ matrices $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in M_{(m+1) \times (m+1)}$, $B \in M_{(m+1) \times (n+1)}$, $C \in M_{(n+1) \times (m+1)}$, and $D \in M_{(n+1) \times (n+1)}$, satisfying the zero supertrace condition $str(x) = tr(A) - tr(D) = 0$. Here, $M_{p \times q}$ denotes the space of all $p \times q$ complex matrices. Let $G_{\bar{0}} = \{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\}$, $G_{\bar{1}} = \{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\}$, then $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is a $\mathbb{Z}_2 (= \mathbb{Z}/2\mathbb{Z})$ graded space over \mathbb{C} with even part $G_{\bar{0}}$ and odd part $G_{\bar{1}}$. G is a Lie superalgebra with respect to the bracket relation defined in the above matrix representation by $[x, y] = xy - (-1)^{ab}yx$, for $x \in G_a$, $y \in G_b$, $a, b \in \mathbb{Z}_2$. $G_{\bar{0}}$ is a Lie algebra isomorphic to $sl(m+1) \oplus \mathbb{C} \oplus sl(n+1)$. Let $G_{+1} = \{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\}$, $G_{-1} = \{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\}$. Then G has a \mathbb{Z}_2 -consistent \mathbb{Z} -grading $G = G_{-1} \oplus G_0 \oplus G_{+1}$, $G_{\bar{0}} = G_0$ and $G_{\bar{1}} = G_{-1} \oplus G_{+1}$.

A Cartan subalgebra H of G has dimension $(m+n+1)$ and consists of diagonal $(m+n+2) \times (m+n+2)$ matrices of zero supertrace, with basis $\{h_i = E_{m+i+1, m+i+1} - E_{m+i+2, m+i+2} \mid i \neq 0, -m \leq i \leq n\} \cup \{h_0 = E_{m+1, m+1} + E_{m+2, m+2}\}$, where E_{ij} is the matrix with 1 in (i, j) -entry and 0 otherwise. The weight space H^* is the dual space of H with basis consisting of the simple roots $\{\alpha_i \mid -m \leq i \leq n\}$ such that the Chevalley generators are $\{e_i = E_{m+i+1, m+i+2}, f_i = E_{m+i+2, m+i+1} \mid -m \leq i \leq n\}$, and

$$\Delta^+ = \{\alpha_{ij} = \sum_{k=i}^j \alpha_k \mid -m \leq i \leq j \leq n\},$$

$$\Delta_0^+ = \{\alpha_{ij} \mid -m \leq i \leq j < 0 \text{ or } 0 < i \leq j \leq n\}, \quad \Delta_1^+ = \{\alpha_{ij} \mid -m \leq i \leq 0 \leq j \leq n\},$$

are respectively the sets of positive roots, positive even roots and positive odd roots. For $\alpha_{ij} \in \Delta^+$, $e_{ij} = E_{m+i+1, m+j+1}$, $f_{ij} = E_{m+j+1, m+i+1}$ are the generators of the root spaces $G_{\alpha_{ij}}$, $G_{-\alpha_{ij}}$ respectively.

Let $(.,.)$ be the inner products in H^* such that $(\alpha_i, \alpha_j) = 2$ if $i = j < 0$; $= -2$ if $i = j > 0$; $= 0$ if $i = j = 0$; $= -1$ if $|i - j| = 1$ and $i, j \leq 0$; $= 1$ if $|i - j| = 1$ and $i, j \geq 0$; and $= 0$ if $|i - j| \geq 2$. Define $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$, $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha$ and $\rho = \rho_0 - \rho_1$. We give a well order in H^* : for $\lambda, \mu \in H^*$, $\lambda > \mu \Leftrightarrow \lambda - \mu = \sum_{i=-m}^n a_i \alpha_i$ such that for the first $a_i \neq 0$, we have $a_i > 0$.

Now let Λ be an integral dominant weight over G , i.e., $\Lambda(h_i) \in \mathbb{Z}_+$ if $i \neq 0$. Let $V^0(\Lambda)$ be the simple highest weight G_0 -modules with the highest weight Λ . As in Ref. 4, we first give some definitions and preliminary results.

Definition 2.1. Extend $V^0(\Lambda)$ to a $G_0 \oplus G_1$ -module by requiring $G_1 V^0(\Lambda) = 0$, and define *Kac-module* to be the induced module $\overline{V}(\Lambda) = \text{Ind}_{G_0 \oplus G_1}^G V^0(\Lambda) = U(G) \otimes_{U(G_0 \oplus G_1)} V^0(\Lambda) \cong U(G_{-1}) \otimes V^0(\Lambda)$. Similarly, one can define the anti-Kac-module $\overline{V}_*(-\Lambda)$ with the lowest weight $-\Lambda$ in the obvious way, starting from the lowest weight G_0 -module $V_*^0(-\Lambda)$ with the lowest weight $-\Lambda$. ■

It follows that any highest weight module with highest weight Λ is a quotient of $\overline{V}(\Lambda)$. We will denote $V(\Lambda)$ the simple highest weight module with highest weight Λ .

Definition 2.2. Let Λ be integral dominant. $\Lambda, \overline{V}(\Lambda), V(\Lambda)$ are called *typical* if there does not exist an odd root $\alpha_{ij} \in \Delta_1^+$ such that $(\Lambda + \rho, \alpha_{ij}) = \sum_{k=i}^0 \Lambda(h_k) - \sum_{k=1}^j \Lambda(h_k) - i - j = 0$, otherwise they are called *atypical*. They are called *singly atypical* if there exists exactly one such α_{ij} (in this case, α_{ij} is called an *atypical root* of Λ), otherwise they are called *multiply atypical* (see Refs. 1, 2 and 3). They are called anti-typical, singly (multiply) anti-atypical if $-\Lambda$ is typical, singly (multiply) atypical respectively. ■

Definition 2.3. A weight vector $v_\lambda \neq 0$ in a module V is called *primitive* (Ref. 5, §9.3) if there exists a submodule U such that $v_\lambda \notin U$ but $G_+ v_\lambda \subset U$. If $U = 0$, i.e., $G_+ v_\lambda = 0$, then v_λ is called *strongly primitive*. Correspondingly, λ is called primitive, or strongly primitive. Denote by P_V the set of primitive weights of V . Similarly, one can define anti-primitive vectors and weights by replacing G_+ by G_- in the definition. ■

In this paper, (anti-) primitive vectors are restricted to be those which are G_0 -strongly (anti-) primitive and which generate indecomposable submodules; different primitive vectors always means they generate different submodules.

In this paper, we will only consider those finite dimensional indecomposable modules with a condition that there exists at least a primitive weight of typical or singly atypical type (note that such condition does not necessarily imply that all primitive weights are typical or singly atypical, we will prove this implication in Theorem 2.9). As H does not always act diagonally on a G -module, in Sects. II and III, we will first consider weight modules V , i.e., V admits a weight space decomposition:

$V = \oplus_{\lambda \in H^*} V_\lambda$, where $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \text{ for } h \in H\}$. (In Ref. 4, V is called a module with diagonal Cartan subalgebra.) Then in Sect. IV, we will extend our results to generalized weight modules.

Lemma 2.4. (1) $\overline{V}(\Lambda)$ is simple $\Leftrightarrow \Lambda$ is typical.

(2) Suppose $\overline{V}(\Lambda)$ is not simple, then (i) if Λ is singly atypical, $\overline{V}(\Lambda)$ has two composition factors: one is $V(\Lambda)$, the other we will denote by $V(\Lambda^-)$; (ii) if Λ is multiply atypical, all primitive weights of $\overline{V}(\Lambda)$ are multiply atypical.

Proof. (1) See Ref. 1. (2)(i) See Refs. 2 and 3. (ii) (See Ref. 3) If Λ is multiply atypical, then the bottom composition factor of $\overline{V}(\Lambda)$ is the only simple submodule of $\overline{V}(\Lambda)$, and its highest weight is multiply atypical. By (i), a highest weight module of singly atypical type (which has at most two composition factors) does not contain such a submodule. Thus a simple highest weight module of singly atypical type cannot be a composition factor of $\overline{V}(\Lambda)$. ■

Remark 2.5. If Λ is singly atypical, by introducing in Ref. 3 the *atypicality matrix* $A(\Lambda)$, an $(m+1) \times (n+1)$ matrix with $(m+i+1, j+1)$ -entry being $A(\Lambda)_{ij} = (\Lambda + \rho, \alpha_{ij})$, and the southwest chain of $A(\Lambda)$, one can obtain Λ^- by subtracting from Λ those α_{ij} sitting on the chain. This can be simply done as follows: choose $\Lambda_0 = \Lambda$, $\Lambda_1 = \Lambda_0 - \alpha_{ij}$, where α_{ij} is the atypical root of Λ , i.e., $A(\Lambda_0)_{ij} = 0$; suppose we have chosen $\Lambda_k = \Lambda_{k-1} - \alpha_{ij}$, whose atypicality matrix $A(\Lambda_k)$ is obtained from $A(\Lambda_{k-1})$ by subtracting 1 from $(m+i+1)$ -th row and adding 1 to $(j+1)$ -th column, then $\Lambda_{k+1} = \Lambda_k - \alpha_{i'j'}$, where $(i', j') \neq (i, j)$ is the another entry satisfying $A(\Lambda_k)_{i'j'} = 0$; continue this procedure until there is no more such $(i'j')$, then Λ_k is dominant and $\Lambda^- = \Lambda_k$. ■

Suppose Λ is singly atypical. We will use the following notations through the paper.

Definition 2.6. (1) Let Λ^- denote the weight defined by Lemma 2.4. Suppose $v_\Lambda, v_{\Lambda^-} \in \overline{V}(\Lambda)$ are the primitive vectors, then there exists $g \in G_{-1}U(G^-)$ such that $gv_\Lambda = v_{\Lambda^-}$. Fix such a g and denote it by g_Λ^- .

(2) Let Λ_{low} denote the lowest weight in $V(\Lambda)$.

(3) Let Λ^+ denote the highest weight of the anti-Kac-module $\overline{V}_*(\Lambda_{low})$. (It follows that Λ_{low} is the lowest weight of the Kac-module $\overline{V}(\Lambda^+)$.)

(4) In anti-Kac-module $\overline{V}_*(\Lambda_{low})$, where Λ_{low} is a singly anti-atypical lowest weight, similar to Lemma 2.4(2.i), there are 2 primitive vectors v_Λ, v_{Λ^+} in $\overline{V}_*(\Lambda_{low})$. By decomposing the enveloping algebra $U(G) = U(G^-)U(G_{+1})U(G_0^+)$, and by noting that v_{Λ^+} is the highest weight vector in $\overline{V}_*(\Lambda)$, there exists $g \in G_{+1}U(G_{+1})$ such that $gv_\Lambda = v_{\Lambda^+}$. Fix such a g and denote it by g_Λ^+ . ■

By the above definition, we see from Refs. 2 and 3 that $\Lambda = (\Lambda^+)^- = (\Lambda^-)^+$ and we can compute

Λ^+ as in Remark 2.5 by defining the northeast chain of $A(\Lambda)$ and adding to Λ those α_{ij} sitting on the chain, so that Λ^+ is the last $\Lambda_k = \Lambda_{k-1} + \alpha_{i'j'}$. Thus, for a singly atypical weight Λ , we can define inductively $\{\Lambda^{(i)} \mid i \in \mathbb{Z}\}$ by $\Lambda^{(0)} = \Lambda$, $\Lambda^{(-i)} = (\Lambda^{(-i+1)})^-$, $\Lambda^{(i)} = (\Lambda^{(i-1)})^+$, $i > 0$. We have $(\Lambda^{(k)})^+ = \Lambda^{(k+1)}$ for $k \in \mathbb{Z}$. For each Λ , denote $\phi_\Lambda = \{\Lambda^{(i)} \mid i \in \mathbb{Z}\}$, and for $i, j \in \mathbb{Z}$, $i \leq j$, denote $\phi_\Lambda^{(ij)} = \{\Lambda^{(k)} \mid i \leq k \leq j\}$.

In the following, we will see a module is uniquely determined, up to an isomorphism, by the relationship between its primitive vectors. We define a diagram to express the structure of a module V , where two primitive vectors v_λ, v_μ are linked by a line with an arrow:

$$(i) v_\lambda \rightarrow v_\mu \Leftrightarrow v_\mu \in G_{-1}U(G^-)v_\lambda \text{ and } (ii) v_\lambda \leftarrow v_\mu \Leftrightarrow v_\lambda \in G_{+1}U(G_{+1})v_\mu. \quad (2.1)$$

It follows that a primitive vector v_λ can be linked by 4 ways: $\rightarrow v_\lambda, \leftarrow v_\lambda, v_\lambda \rightarrow, v_\lambda \leftarrow$.

Definition 2.7. For a module V , let W_V be a set of primitive vectors of V corresponding to a composition series. We can associate W_V with a diagram defined by (2.1) for $v_\lambda, v_\mu \in W_V$. We call this diagram the *weight diagram* of V , and denote it again by W_V . ■

From this definition, we see that the weight diagram depends on the choices of primitive vectors: a module V may correspond to more than one weight diagrams. However, a weight diagram W_V does determine the structure of V as we will see later. When there is no confusion, we sometimes may use V to mean its diagram or vice versa.

Definition 2.8 A *cyclic* module $X(\Lambda)$ is a module generated by a primitive vector v_Λ . ■

Theorem 2.9. Suppose V is an indecomposable module with a primitive weight Λ and a primitive vector v_Λ . We have

- (1) If Λ is typical, then $V = \overline{V}(\Lambda)$.
- (2) If Λ is multiply atypical, then all primitive weights are multiply atypical.
- (3) If Λ is singly atypical, then all the primitive weights are singly atypical and $P_V = \phi_\Lambda^{(ij)}$ for some $i \leq 0 \leq j$.
- (4) For any choice of W_V , W_V must be connected, i.e., for any $u, v \in W_V$, there exist $u_0 = u, u_1, \dots, u_k = v \in W_V$ for some k such that u_i is linked to u_{i+1} by a line with an arrow for $i = 0, \dots, k-1$.

Proof. Take a composition series: $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$. We will prove the result by induction on n . If $n \leq 2$, by Lemma 2.4, we have the result. Now suppose $n \geq 3$. We first prove *Statement (A)*: All primitive weights must have the same type: typical, singly atypical or multiply atypical, and W_V must be connected. *Case (a)*: V/V_1 is decomposable. Decompose it into a direct sum of indecomposable submodules: $V/V_1 = \bigoplus_{i=1}^k V'_i/V_1$, then each $V'_i \supset V_1$ must be indecomposable (if $V'_i = V''_i \oplus V_1$, we would then write V as a disjoint sum $V''_i \oplus (\sum_{j \neq i} V'_j)$), and by inductive assumption,

for all i , all primitive weights of V'_i , which contains V_1 , have the same type and $W_{V'_i}$ is connected, and thus, all primitive weights of V have the same type as that of V_1 and W_V is connected. *Case (b):* V/V_1 is indecomposable. By inductive assumption, all primitive weights in V/V_1 are of the same type and W_{V/V_1} is connected. This means that $(n-1)$ of n primitive weights of V are of the same type. If V_1 is the only simple submodule of V , then V_{n-1} is also indecomposable, and so by inductive assumption, primitive weights of V_{n-1} have the same type and $W_{V_{n-1}}$ is connected; or else, if V has another simple submodule V'_1 , then again by inductive assumption, primitive weights of V/V'_1 have the same type and W_{V/V'_1} is connected. In either aspect, we can find another primitive vector of V linking to, and having the same type with, the primitive vector of V_1 . Therefore all primitive weights of V have the same type and W_V is connected. This proves Statement (A), which implies (2)&(4).

Now (1) can be proved as above by induction on n . To complete the proof of (3), by Statement (A), we see that all primitive weights are now singly atypical. We divide the proof into three steps: *Step (i):* Suppose V is cyclic. Take $U_1 = V/V_1$ and take $U_2 = V_2$ if V_1 is the only simple submodule of V , or else, $U_2 = V/V'_1$ if V'_1 is another simple submodule V'_1 of V . Then in either case, U_1 and U_2 are both cyclic. By inductive assumption, $P_{U_1} = \phi_\lambda^{(ij)}$, $P_{U_2} = \phi_\mu^{(i'j')}$ for some λ, μ , and P_V is their union. As they are not disjoint, and Λ is at least in one of them. We see that P_V has the required form. *Step (ii):* For a primitive weight λ , let $V^{(\lambda)}$ be the submodule generated by primitive vectors with weights in ϕ_λ . By Step (i), $P_{V^{(\lambda)}} \subset \phi_\lambda$. For any two weights λ, μ , ϕ_λ and ϕ_μ are either the same set or they are disjoint, thus different $V^{(\lambda)}$ are disjoint. Since V , being a disjoint sum of $V^{(\lambda)}$, is indecomposable, we must have $V = V^{(\Lambda)}$. *Step (iii):* Suppose $\Lambda^{(k)} \notin P_V$ but $\Lambda^{(i)}, \Lambda^{(j)} \in P_V$ for some $i < k < j$. Let W_1 and W_2 be submodules generated by primitive vectors with weight $\Lambda^{(r)}$ such that $r > k$ and $r < k$ respectively. Then $V = W_1 \oplus W_2$, a contradiction with that V is indecomposable. Thus we have (3). ■

Remark 2.10. (1) If the weight diagram W_V is connected, it is not necessary that V is indecomposable (see Remark 3.7). However, if V is decomposable, we can always choose some suitable primitive vectors such that W_V is not connected.

(2) $v_\mu \in U(G)v_\lambda$ does not mean that v_μ is linked to v_λ as we will see in Lemma 3.5(1.v). ■

III. INDECOMPOSABLE WEIGHT MODULES

Now we can classify indecomposable modules V with a primitive weight Λ of typical or singly atypical type. We do this by classifying the weight diagram. If Λ is typical, then by Theorem 2.9, V is simply $\overline{V}(\Lambda)$. Thus, from now on, we suppose Λ is singly atypical. Then, all primitive weights are

singly atypical (again by Theorem 2.9).

Lemma 3.1. For any primitive vector v_λ , there exists at most one primitive vectors v_μ such that $\mu < \lambda$ and $v_\mu \in U(G)v_\lambda$ (or $\mu > \lambda$ and $v_\mu \in U(G)v_\lambda$).

Proof. Suppose conversely there exists a cyclic module $V_1 = U(G)v_\lambda$ of the lowest dimension such that there are 2 primitive vectors $v_\mu, v_\sigma \in V_1$ with, say, $\mu, \sigma < \lambda$. By Theorem 2.9(3), $\mu, \sigma \leq \lambda^-$. If there is a primitive vector $v_\delta \in V_1$ such that $\delta > \lambda$ (and then $\delta \geq \lambda^+$), then by our choice of V_1 being lowest dimensional, $V'_1 = U(G)v_\delta$ does not have more than one primitive weight $\delta_1 < \delta$. Hence, by Theorem 2.9(3), if $\delta_1 < \delta$ is a primitive weight of V'_1 , then $\delta_1 = \delta^- > \lambda^- \geq \mu, \sigma$. Thus, if we let V_2 be the module generated by $\{v_\delta \mid \delta > \lambda, v_\delta \in V_1 \text{ primitive}\}$, then μ, σ are not primitive weights in V_2 , so v_μ, v_σ are still primitive in $V_3 = V_1/V_2$. Since V_3 is also a cyclic module, by our choice of V_1 , we must have $V_2 = 0$, i.e., V_1 is a highest weight module, but by Lemma 2.4(2.i), V_1 cannot contain 2 primitive weights μ, σ . Thus we obtain a contradiction. ■

Corollary 3.2. W_V does not contain (1) $u \rightarrow v \rightarrow w$, (2) $u \xrightarrow{v} w$, (3) $u \leftarrow v \leftarrow w$, or (4) $u \xleftarrow{v} w$.

Proof. This follows immediately from Lemma 3.1 (say V_W contains (1) or (2), then $v, w \in U(G)u$ and their weights are less than the weight of v). ■

Corollary 3.3. (1) If $v_\lambda \rightarrow v_\mu$ or $v_\lambda \leftarrow v_\mu$, then $\lambda = \mu^+$. (2) $P_{U(G)v_\lambda} \subset \{\lambda^+, \lambda, \lambda^-\}$.

Proof. (1) Say, $v_\lambda \rightarrow v_\mu$. Let $V_1 = U(G)v_\lambda$. By Theorem 2.9(3), if $\mu \neq \lambda^-$, then $\mu \leq \lambda^{(-2)}$, and λ^- must also be a primitive weight of V_1 . This contradicts Lemma 3.1. (2) follows from (1) and Lemma 3.1. ■

Lemma 3.4. If (1) $v_{\Lambda^+} \xleftarrow{v_\Lambda} u_\Lambda$ or (2) $u_\Lambda \xrightarrow{v_\Lambda} v_{\Lambda^-}$ is a part of the diagram, then W_V must contain $X_4(\Lambda)$: $v_{\Lambda^+} \xleftarrow{v_\Lambda} u_\Lambda \xrightarrow{v_\Lambda} v_{\Lambda^-}$.

Proof. Suppose conversely $V_1 = U(G)v_\Lambda$ is cyclic with the lowest dimension such that its weight diagram contains, say, (2), but does not contain $X_4(\Lambda)$. By Lemma 2.4(2.i), v_Λ is not strongly primitive, i.e., there exists v_{Λ^+} such that $v_{\Lambda^+} \leftarrow v_\Lambda$. By Corollary 3.2, we do not have a primitive vector x such that $x \leftarrow v_{\Lambda^+} \leftarrow v_\Lambda$, so v_{Λ^+} must be strongly primitive. Let $V_2 = U(G)v_{\Lambda^+}$. If $u_\Lambda \notin V_2$, then the weight diagram of V_1/V_2 contains $u_\Lambda \xrightarrow{v_\Lambda} v_{\Lambda^-}$ which violates the choice of V_1 being of lowest dimension. Thus $u_\Lambda \in V_2$, i.e., $u_\Lambda \in G_{-1}U(G_{-1})v_{\Lambda^+}$, i.e., $v_{\Lambda^+} \rightarrow u_\Lambda$, a contradiction. ■

Lemma 3.5. If $V = X(\Lambda)$ is cyclic, then

- (1) W_V is one of (i) v_Λ , (ii) $v_{\Lambda^+} \leftarrow v_\Lambda$, (iii) $v_\Lambda \rightarrow v_{\Lambda^-}$, (iv) $v_{\Lambda^+} \leftarrow v_\Lambda \rightarrow v_{\Lambda^-}$, or (v) $X_4(\Lambda)$. $X_4(\Lambda)$ is the only cyclic module containing a primitive vector which is not linked by the generator v_Λ .
- (2) V is uniquely (up to isomorphisms) determined by its diagram.

Proof. (1) If v_Λ is not linked to any primitive vector on one side, then V is a quotient of Kac- (or anti-Kac-) module, and we have one of (i), (ii) or (iii). Suppose now v_Λ is linked to primitive vectors on both sides, then W_V contains (iv). If it is not (iv), then there is another primitive vector u , by Lemma 3.1, it must have weight Λ (and we denote $u_\Lambda = u$). Since both modules $V/U(G)v_{\Lambda+}$, $V/U(G)v_{\Lambda-}$ are Kac- or anti-Kac- modules, we must have $u_\Lambda \in U(G)v_{\Lambda+}$ and $u_\Lambda \in U(G)v_{\Lambda-}$, i.e., $v_{\Lambda+}$ and $v_{\Lambda-}$ are both linked to u_Λ with an arrow pointed to u_Λ . By Corollary 3.2, $u = u_\Lambda$ is unique, i.e., we have (v). The statement about $X_4(\Lambda)$ is clearly true.

(2) Let g_Λ^+, g_Λ^- be as in Definition 2.6, if necessary, by replacing $v_{\Lambda+}$ by $g_\Lambda^+ v_\Lambda$, $v_{\Lambda-}$ by $g_\Lambda^- v_\Lambda$, we can suppose $v_{\Lambda+} = g_\Lambda^+ v_\Lambda$, $v_{\Lambda-} = g_\Lambda^- v_\Lambda$. This uniquely determines V in the first 4 cases. Suppose now V is (v), we can choose $u_\Lambda = g_{\Lambda+}^- v_{\Lambda+}$ and $g_{\Lambda-}^+ v_{\Lambda-}$ must be a nonzero multiple of u_Λ , i.e., $g_{\Lambda-}^+ v_{\Lambda-} = xu_\Lambda$ for some $0 \neq x \in \mathbb{C}$. Suppose V' is another module with weight diagram (v) (denote its corresponding primitive vectors by the same notation with a prime) and suppose $g_{\Lambda-}^+ v'_{\Lambda-} = yu'_\Lambda$ for some $0 \neq y \in \mathbb{C}$, $y \neq x$. Form a direct sum $V \oplus V'$ and let V'' be its submodule generated by $v''_\Lambda = v_\Lambda + v'_\Lambda$. Then V'' is indecomposable, which contains primitive vectors $v''_{\Lambda+} = v_{\Lambda+} + v'_{\Lambda+}$, $v''_{\Lambda-} = v_{\Lambda-} + v'_{\Lambda-}$, $u''_\Lambda = u_\Lambda + u'_\Lambda$ and $u'''_\Lambda = xu_\Lambda + yu'_\Lambda$. This contradicts (1). (We will see from (4.11) that $x = -1$.) ■

In the following, to be consistent, we use the same notations as in Ref. 4.

Theorem 3.6. (1) W_V must be one of the following: (i) $X_4(\Lambda)$,
(ii)(a) $X_{5a}(\Lambda, n) : v_\Lambda \rightarrow v_1 \leftarrow v_2 \rightarrow \dots$ (ended by $v_{n-1} \rightarrow v_n$, or $v_{n-1} \leftarrow v_n$ if n is odd or even),
(b) $X_{5b}(\Lambda, n) : v_\Lambda \leftarrow v_1 \rightarrow v_2 \leftarrow \dots$ (ended by $v_{n-1} \leftarrow v_n$ or $v_{n-1} \rightarrow v_n$ if n is odd or even),
where v_i has weight $\Lambda^{(-i)}$.

(2) V is uniquely determined up to isomorphisms by W_V .

Proof. For (2), Lemma 3.5(2) tells that $X_4(\Lambda)$ determines V . As in the proof of Lemma 3.5(2), $X_{5a}(\Lambda, n)$, $X_{5b}(\Lambda, n)$ also uniquely determine V . In fact, we can choose v_i inductively, such that $v_0 = v_\Lambda$, $v_{i-1} = g_{\Lambda(-i)}^+ v_i$ if $v_{i-1} \leftarrow v_i$ and $g_{\Lambda(-i+1)}^- v_{i-1} = v_i$ if $v_{i-1} \rightarrow v_i$.

(1) (cf. proof of Theorem 4.9(2.ii).) If V has a submodule corresponds to (i), denote it by V_1 ; otherwise, let V_1 be a maximal submodule whose diagram is (ii)(a) or (ii)(b). If V has more primitive vectors, choose B to be its other primitive vectors such that $\sum_{v \in B} \dim(V_1 \cap U(G)v)$ is minimum. If B is not empty, by Theorem 2.9(4), there exists $v \in B$ linking to V_1 . (i) If $V_1 = X_4(\Lambda)$. By Corollary 3.2, the only possible links are: (a) $v_{\Lambda+} \leftarrow v$, (b) $v \rightarrow v_{\Lambda-}$, (c) $v_{\Lambda+} \leftarrow v \rightarrow v_{\Lambda-}$, (d) $v \rightarrow u_\Lambda$, (e) $u_\Lambda \leftarrow v$. For the first three cases, v has weight Λ . Say, we have (c). We can choose v such that $v_{\Lambda+} = g_\Lambda^+ v$ and $g_\Lambda^- v = xv_{\Lambda-}$, for some $x \neq 0$. Apply Lemma 3.5(2) to $U(G)v$, whose diagram is $X_4(\Lambda)$, we must have $x = 1$. Thus if we let $v' = v - v_\Lambda$, then $V_1 \cap U(G)v' = 0$. By replacing v by v' in B , we get a contradiction with the choice of B . Similarly, for other cases, we can also choose some

primitive vector $v'' \in V_1$ such that if we replace v by $v - v''$ we obtain a contradiction with the choice of B . Therefore, B is empty, and $V = X_4(\Lambda)$. (ii) As V_1 is maximal, by Lemma 3.4 and Corollary 3.2, v can not be linked to v_Λ or v_n . Also by Corollary 3.2, v can not be linked to a vector v_{2i} of $X_{5a}(\Lambda, n)$, or v_{2i-1} of $X_{5b}(\Lambda, n)$. On the other hand, if v is linked to a vector $v_{2i-1} \in X_{5a}(\Lambda, n)$ or $v_{2i} \in X_{5b}(\Lambda, n)$, then v must be linked to that vector with an arrow pointed to it. Then as in (i), by replacing v by $v - v''$ for some $v'' \in V_1$, we can get a contradiction. Thus again, B is empty and we have (ii). \blacksquare

Remark 3.7. Diagrams such as $u \xleftarrow{v} w$ and $u \xrightarrow{v} w$ can exist, but they correspond to decomposable modules: by replacing w by $w - v$, we see that w is not linked to u , v (see Remark 2.10). \blacksquare

It is not difficult to construct $X_{5a}(\Lambda, n)$, $X_{5b}(\Lambda, n)$ as follows: in the module $\overline{V}_*((\Lambda^-)_{low}) \oplus \overline{V}(\Lambda^-)$, whose diagram has two parts: $\leftarrow v'_{\Lambda^-}, v''_{\Lambda^-} \rightarrow$, “joining” two primitive vectors $v'_{\Lambda^-}, v''_{\Lambda^-}$ into one, by letting $v_{\Lambda^-} = v'_{\Lambda^-} + v''_{\Lambda^-}$, we obtain $X_{5b}(\Lambda, 3) = U(G)v_{\Lambda^-}$. Now $X_{5b}(\Lambda^{(-2i)}, 3)$ has diagram $v'_{\Lambda^{(-2i)}} \leftarrow v_{\Lambda^{(-2i-1)}} \rightarrow v''_{\Lambda^{(-2i-2)}}$. By taking a quotient module, by “merging” $v'_{\Lambda^{(-2i)}}, v''_{\Lambda^{(-2i)}}$ into one, we obtain $X_{5b}(\Lambda, 2n) = \oplus_{i=0}^n X_{5b}(\Lambda^{(-2i)}, 3) / \oplus_{i=1}^n U(G)(v'_{\Lambda^{(-2i)}} - v''_{\Lambda^{(-2i)}})$. Modules $X_{5a}(\Lambda, n)$, $X_{5b}(\Lambda, 2n+1)$ can be realized as subquotients of $X_{5b}(\Lambda, k)$ for some k . To construct $X_4(\Lambda)$, form an induced module $\widehat{V}(\Lambda) = \text{Ind}_{G_0}^G V^0(\Lambda) = U(G) \otimes_{U(G_0)} V^0(\Lambda) \cong U(G_{-1}) \otimes U(G_{+1}) \otimes V^0(\Lambda)$. (Note that this module is in general decomposable, therefore not cyclic. However a cyclic module can be realized as its quotient module.) We are not going to realize $X_4(\Lambda)$ to be a quotient module of $\widehat{V}(\Lambda)$, but as a submodule of $\widehat{V}(\Lambda^+ - 2\rho_1)$: let $v_{\Lambda^+ - 2\rho_1}$ be the highest weight vector in G_0 -module $V^0(\Lambda^+ - 2\rho_1)$, then $v_{\Lambda^+} = gv_{\Lambda^+ - 2\rho_1}$ (where g is the highest root vector in $U(G_{+1})$) is a strongly primitive vector with weight Λ^+ , which generates Kac-module $\overline{V}(\Lambda^+)$. It is clear that there must exist a primitive vector v_Λ in $\widehat{V}(\Lambda^+ - 2\rho_1)$ such that $v_{\Lambda^+} \leftarrow v_\Lambda$. By Lemma 3.4 and Theorem 3.6, v_Λ generates a module corresponding to $X_4(\Lambda)$ (see also (4.11)).

We see that just as an anti-Kac-module is isomorphic to a Kac-module, $X_{5b}(\Lambda, 2n+1)$ is isomorphic to $X_{5a}(\Sigma, 2n+1)$ for some Σ (in fact, $(\Sigma^{(-2n-1)})_{low} = -\Lambda$). To see that $X_4(\Lambda)$, $X_{5a}(\Lambda, n)$, $X_{5b}(\Lambda, n)$ are indecomposable: suppose $V = V_1 \oplus V_2$ is a disjoint sum, then each simple submodule must be contained in V_1 or V_2 , and then we can obtain that all primitive vectors must be in one, say, V_1 , and $V = V_1$. Now we can conclude the following

Theorem 3.8. $\{\overline{V}(\Lambda) \mid \Lambda \text{ typical}\} \cup \{X_4(\Lambda), X_{5a}(\Lambda, n), X_{5b}(\Lambda, 2n) \mid n \in \mathbb{Z}_+ \setminus \{0\}\}$ is the complete set of indecomposable modules with at least a primitive weight of typical or singly atypical type.

Proof. It remains to prove there is no isomorphism between each other. This can be seen by comparing number of simple submodules and number of composition factors. \blacksquare

IV. INDECOMPOSABLE GENERALIZED WEIGHT MODULES

Now suppose V is an indecomposable G -module such that H acts on V not necessarily diagonally. Such a module is called a generalized weight module (in Ref. 4, it is called a module with nondiagonal Cartan subalgebra, or nondiagonal module). In this case, we do not have weight space decomposition. However, by the properties of semi-simple Lie algebras, we see that V must be H_0 -diagonal, where H_0 is the Cartan subalgebra of G'_0 (where, here and after, G'_0 is the subalgebra of G_0 with co-dimension 1 such that $h_0 \notin G'_0$) with a basis $\{h_i \mid i \neq 0\}$. We have

$$V = \oplus_{\lambda \in H^*} \mathbf{V}_\lambda, \quad \text{where } \mathbf{V}_\lambda = \{v \in V \mid (h - \lambda(h))^n v = 0 \text{ for some } n \in \mathbb{Z}_+\}. \quad (4.1)$$

\mathbf{V}_λ are called *generalized* weight spaces. With this decomposition, we have, similar to Sect. II, notions of *generalized* (strongly) (anti-) primitive vectors (weights), *generalized* weight diagram, etc. In the following, we often omit the word *generalized* if there is no confusion.

We can take a composition series

$$0 = V^{(0)} \subset V^{(1)} \subset \dots \subset V^{(n)} = V, \quad (4.2)$$

such that each $V^{(i)}$ is a direct sum of subspaces

$$V^{(i)} = V^{(i-1)} \oplus \overline{V}^{(i)}, \quad \text{where } \overline{V}^{(i)} \text{ is a } G'_0\text{-module.} \quad (4.3)$$

Then each $\overline{V}^{(i)}$ has a unique, up to scalars, generalized primitive vector $v_{\lambda_i}^{(i)}$ with weight λ_i . Sometimes, we can just choose (4.2) to be any series of submodules such that (4.3) holds.

Definition 4.1. For an integral dominant weight Λ , construct an indecomposable module, the *generalized Kac-module* $\overline{V}(\Lambda, n)$, $n \in \mathbb{Z}_+ \setminus \{0\}$, as follows: It is a semi-direct sum of n copies of $\overline{V}(\Lambda)$, such that each copy is a $G'_0 \oplus G_{-1}$ -module, and

$$h_0 v_\Lambda^{(i)} = \Lambda(h_0) v_\Lambda^{(i)} + v_\Lambda^{(i-1)}, \quad i = 2, \dots, n, \quad (4.4)$$

where $v_\Lambda^{(i)}$ belongs to the i -th copy of $\overline{V}(\Lambda)$. As $\overline{V}(\Lambda)$ is the induced module, it is easy to check that $\overline{V}(\Lambda, n)$ is well defined as an indecomposable G -module. Similarly, we can define generalized anti-Kac-module $\overline{V}_*(-\Lambda, n)$. ■

Note that if Λ is singly atypical, then $\overline{V}(\Lambda, n)$ has the “zigzag” weight diagram such that

$$v_\Lambda^{(i)} \rightarrow v_{\Lambda^-}^{(i)}, \quad v_{\Lambda^-}^{(i-1)} \leftarrow v_{\Lambda^-}^{(i)}, \quad (4.5)$$

for $i = n, n-1, \dots, 1$ (where we take $v_\Lambda^{(0)} = 0$).

Remark 4.2. If Λ is typical, $\overline{V}(\Lambda, n)$ give us examples that there may exist a primitive vector v_Λ such that it is not linked to any primitive vector, but $U(G)v_\Lambda$ is not simple and that an indecomposable module may not correspond to a connected weight diagram. Therefore, more care should be taken when we consider generalized weight modules. We shall see in Lemmas 4.5(2)&4.6(3), that this does not happen if Λ is singly atypical. ■

Definition 4.3. Define a partial order on generalized primitive vectors: we say v has *higher level* than u or v is on *top* of $u \Leftrightarrow u \in U(G)v$, but $v \notin U(G)u$. ■

We see that the generalized primitive vectors of $\overline{V}_0(\Lambda, n) = \overline{V}(\Lambda, n)$ are well ordered by this partial order. Now by removing top vector $v_\Lambda^{(n)}$, removing bottom vector $v_{\Lambda^-}^{(1)}$, and removing both top and bottom vectors $v_\Lambda^{(n)}, v_{\Lambda^-}^{(1)}$ respectively, we obtain three indecomposable modules: $\overline{V}_1(\Lambda, n) = U(G)v_{\Lambda^-}^{(n)}$, $\overline{V}_2(\Lambda, n) = \overline{V}(\Lambda, n)/U(G)v_{\Lambda^-}^{(1)}$ and $\overline{V}_3(\Lambda, n-1) = U(G)v_{\Lambda^-}^{(n)}/U(G)v_{\Lambda^-}^{(1)}$. We see that generalized anti-Kac-module $\overline{V}_*((\Lambda^-)_{low}, n)$ can be realized as $\overline{V}_3(\Lambda, n)$, a subquotient of generalized Kac-module $\overline{V}(\Lambda, n+1)$.

Lemma 4.4. Suppose V is a (generalized) highest weight module with highest weight Λ (i.e., V is generated by a generalized strongly primitive vector v_Λ). (1) If Λ is typical, then $V \cong \overline{V}(\Lambda, n)$ for some n . (2) If Λ is singly atypical, then V is a quotient of $\overline{V}(\Lambda, n)$ for some n . More precisely, $V = \overline{V}_i(\Lambda, n)$, $i = 0$ or 2 .

Proof. First, as $U(G) = U(G_-)U(H)U(G_+)$, we see that $V = U(G_-)U(H)v_\Lambda$ does not have weight $> \Lambda$. Let V be as in (4.2). We use induction on n . If $n = 1$, the result is obvious. Suppose now $n \geq 2$. As $V' = V/V^{(1)}$ is still a highest weight module, by inductive assumption, it has the required form. If Λ is typical, then $V' = \overline{V}(\Lambda, n-1)$, and therefore by (4.4), we can inductively choose spaces $\overline{V}^{(i)}$ in (4.3) as a copy of $\overline{V}(\Lambda)$ with the primitive vector $v_\Lambda^{(i)}$, $i = n, n-1, \dots, 2$, such that

$$h_0 v_\Lambda^{(i)} = \Lambda(h_0)v_\Lambda^{(i)} + v_\Lambda^{(i-1)} + v_i \text{ for some } v_i \in V^{(1)}, \quad (4.6)$$

where, we take $v_\Lambda^{(1)} = 0$ when $i = 2$. As all $v_\Lambda^{(i)} \in \mathbf{V}_\Lambda$, we have $v_i \in \mathbf{V}_\Lambda$, i.e., v_i has weight Λ . By replacing $v_\Lambda^{(i-1)}$ by $v_\Lambda^{(i-1)} + v_i$ (and replacing space $\overline{V}^{(i)}$ by $U(G_{-1})U(G'_0)(v_\Lambda^{(i-1)} + v_i)$ accordingly), we can suppose $v_i = 0$ if $i > 2$. If $v_2 \neq 0$, denote it by $v_\Lambda^{(1)}$, then we have (4.4), and thus $V = \overline{V}(\Lambda, n)$ and the result follows. On the other hand, if $v_2 = 0$, then $V'' = \sum_{i=2}^n \overline{V}^{(i)}$ is a submodule of V and $V = V^{(1)} \oplus V''$, a contradiction with that V is a highest weight module. This proves (1). For (2), as V' has the required form, without loss of generality, say $V' = \overline{V}_2(\Lambda, n)$. Then as space, it is the

direct sum of $(n - 1)$ copies of $\overline{V}(\Lambda)$ plus $V(\Lambda)$. Now again choose $\overline{V}^{(i)}$ to be a copy of $\overline{V}(\Lambda)$ for $i \geq 3$ and $\overline{V}^{(2)} = V(\Lambda)$ (in this case, (4.2) is not a composition series). Now follow the arguments exactly as above, we have (4.4) for $i \geq 3$. Take $v_{\Lambda}^{(i)}$ to be the other primitive vector in $\overline{V}^{(i)}$ for $i \geq 3$. Then we have (4.5) for $i = n, \dots, 3$. Now we must have $\overline{V}^{(1)} \subset V''' = U(G)v_{\Lambda}^{(2)}$ (otherwise, $V = \overline{V}^{(1)} \oplus (\sum_{i=2}^n \overline{V}^{(i)})$ is decomposable). It remains to prove V''' , which is now as space $\overline{V}^{(1)} \oplus \overline{V}^{(2)}$, is $\overline{V}(\Lambda)$ (and then $V = \overline{V}(\Lambda, n)$). This follows from Lemma 4.5(3) below. ■

The following Lemma 4.5 tells that, unlike typical case, for atypical weight Λ , we do not have indecomposable module whose composition factors are n copies of $V(\Lambda)$.

Lemma 4.5. (1) Suppose V is a (generalized weight) module whose composition factors are n copies of $V(\Lambda)$ with Λ typical or singly atypical, then (i) if V is indecomposable, then either $n = 1$, or else, Λ is typical and $V = \overline{V}(\Lambda, n)$; (ii) if Λ is singly atypical, then V is the direct sum of n copies of $V(\Lambda)$ (and thus, V is a weight module).

(2) If v_{Λ} is a primitive vector with singly atypical weight Λ such that it is not linked to any primitive vector, then $U(G)v_{\Lambda}$ is simple.

(3) If V is an indecomposable module with 2 composition factors of singly atypical type, then V is a Kac- (or anti-Kac-) weight module.

Proof. (1) Let (4.2) be a composition series of V . By induction on n , we see that we only need to prove the result for $n = 2$. (i) If Λ is typical, it is easy to see, as in the proof of Lemma 4.4, $V = \overline{V}(\Lambda, 2)$. (ii) Suppose now Λ is singly atypical, let $v_{\Lambda}^{(i)}$, $i = 1, 2$, be the primitive vectors of V and suppose

$$h_0 v^{(1)} = \Lambda(h_0) v^{(1)}, \quad h_0 v_{\Lambda}^{(2)} = \Lambda(h_0) v_{\Lambda}^{(2)} + a v_{\Lambda}^{(1)}, \quad \text{for some } a \in \mathbb{C}. \quad (4.7)$$

Note that, in $U(G)$, for $G = sl(m/n)$, using notations in Sect. II, we have

$$\prod_{-m \leq i \leq 0 \leq j \leq n} e_{ij} \prod_{-m \leq i \leq 0 \leq j \leq n} f_{ij} = \sigma \prod_{-m \leq i \leq 0 \leq j \leq n} (\Sigma_{k=i}^0 h_k - \Sigma_{k=1}^j h_k - i - j) + g^+, \quad (4.8)$$

for some $g^+ \in U(G)G^+$, where $\sigma = \pm 1$ (this can be proved by ordering e_{ij} (f_{ij}) properly in the products, such that if $j - i > j' - i'$, or $j - i = j' - i'$ and $j > j'$, then e_{ij} (f_{ij}) is placed to the right (left) of $e_{i'j'}$ ($f_{i'j'}$ resp.); then using induction on m, n). We have

$$U(G)G^+ v_{\Lambda}^{(2)} = 0, \quad \prod_{-m \leq i \leq 0 \leq j \leq n} f_{ij} v_{\Lambda}^{(2)} = 0. \quad (4.9)$$

(The l.h.s. of the second equality is in the bottom composition factor $V(\Lambda^-)$ of $\overline{V}(\Lambda)$ with weight $\Lambda - 2\rho_1$, any copy of $V(\Lambda)$ does not have a vector with weight $\Lambda - 2\rho_1$; therefore it is zero in V). Now

apply (4.8) to $v_\Lambda^{(2)}$, using (4.9) and (4.7), we obtain

$$0 = \prod_{-m \leq i \leq 0 \leq n} (\Lambda + \rho, \alpha_{ij}) v_\Lambda^{(2)} + \sum_{-m \leq i \leq 0 \leq j \leq n} \prod_{-m \leq i' \leq 0 \leq j' \leq n, (i'j') \neq (i,j)} (\Lambda + \rho, \alpha_{i'j'}) a v_\Lambda^{(1)}. \quad (4.10)$$

However, by Definition 2.2, there is exactly one atypical root, i.e., one pair of (i, j) such that $(\Lambda + \rho, \alpha_{ij}) = 0$, thus (4.10) forces $a = 0$. This proves $V = U(G)v_\Lambda^{(2)} \oplus U(G)v_\Lambda^{(1)} = V(\Lambda) \oplus V(\Lambda)$.

(2) Let $V = U(G)v_\Lambda$ be as in (4.2). If $n > 1$, using induction, we can suppose $V/V^{(1)}$ is simple, i.e., $n = 2$. By (1), two composition factors can not be the same, but then as in (4.6) (for $i = 2$) and the arguments after (4.6), we see that $U(G)v_\Lambda$ is a weight module. As v_Λ is not linked to any primitive vector, by Theorem 2.9, V has to be simple (and so $n = 2$ does not occur).

(3) The proof is the same as (2). ■

Now we have the generalized weight modules $\bar{V}_i(\Lambda, n)$ and $X_{5a}(\Lambda, n)$, $X_{5b}(\Lambda, n)$, whose diagrams have 2 *endpoints* (i.e., vectors linked by only one vector). Such diagrams are called *lines*. We can “join” and “merge” those modules to form other indecomposable modules just as we did to form $X_{5b}(\Lambda, n)$ in Sect. III. In particular, we can define a module $X_4(\Lambda, m, n, x)$ ($m, n \geq 2$, $0 \neq x \in \mathbb{C}$) as follows: let $X_3(\Lambda, m, n) = (\bar{V}_1(\Lambda^+, m) \oplus \bar{V}_2(\Lambda, n)) / U(G)(v_\Lambda^{(1)} - v_\Lambda'^{(1)})$ (with obvious meanings of notations) be the quotient module (by “merging” the two bottom endpoints $v_\Lambda^{(1)} \in \bar{V}_1(\Lambda^+, m)$ and $v_\Lambda'^{(1)} \in \bar{V}_2(\Lambda, n)$). Then $X_4(\Lambda, m, n, x)$ is the submodule of $X_3(\Lambda, n)$ generated by $v_\Lambda^{(m)} + x v_\Lambda'^{(n)}$ (by “joining” the two top endpoints). By this construction, one sees that $X_4(\Lambda, m, n, x)$ is indecomposable, generated by a primitive vector, therefore cyclic. Its diagram is a *circle*, i.e., no endpoints. It is interesting to see that $X_4(\Lambda)$ can be realized as

$$X_4(\Lambda) = X_4(\Lambda, 2, 2, -1). \quad (4.11)$$

This is because: if $x = -1$, when we “join” the top endpoints, the second term of the *r.h.s.* of (4.4) is lost, and so h_0 becomes acting diagonally. We point out that only with a circle, an x makes difference: just as in Sect. III, we can choose suitable primitive vectors starting from a vector v_{λ_1} of the weight diagram (we always choose v_{λ_1} to be an endpoint if it is a line), and follow the links between vectors, such that if $v_\lambda \rightarrow v_\mu$, then $g_\lambda^- v_\lambda = v_\mu$ and if $v_\lambda \leftarrow v_\mu$, then $v_\lambda = g_\mu^+ v_\mu$. But within a circle, the last vector v_{λ_n} we chose is linked to the first one and in this case, say, $v_{\lambda_n} \leftarrow v_{\lambda_1}$, we may have $g_{\lambda_1}^+ v_{\lambda_1} = x v_{\lambda_n}$ for some $x \in \mathbb{C} \setminus \{0\}$. By rescaling vectors, we see that x can be shifted anywhere, but can not be eliminated if the diagram is a circle.

It is interesting to see that we can “add” a primitive vector to a circle to break it into a line: in the above construction, $X_3(\Lambda, m, n)$ can be obtained by adding $v_\Lambda^{(m)}$ to $X_4(\Lambda, m, n, x)$.

Lemma 4.6. (1) Let V be a cyclic module with a primitive weight Λ of typical or singly atypical type, then all generalized primitive weights have the same type. Furthermore, V must be (i) $\overline{V}(\Lambda, n)$ if Λ is typical or (ii) a quotient of some $X_4(\Sigma, m, n, x)$, where $\Sigma = \Lambda^{(i)}$ for some $i = -1, 0, 1$ (and so its diagram is obtained from $X_4(\Sigma, m, n, x)$ by removing some vectors from the bottom) if Λ is singly atypical.

(2) If Λ is singly atypical and $V = X(\Lambda)$ is cyclic, then $P_V \in \{\Lambda^+, \Lambda, \Lambda^-\}$.

(3) If V is an indecomposable module with a singly atypical weight, then W_V is connected.

Proof. (1) Take a composition series $0 = V_0 \subset V_1 \subset \dots \subset V_k = V$. First as in the proof of Theorem 2.9, by induction on k , we can prove all generalized primitive weights have the same type. Now if $k = 1$, we clearly have the result. Suppose $k \geq 2$. Then V/V_1 is still cyclic, by inductive assumption, it has the required form. Now follows exactly the same arguments as in the proof of Lemma 4.4, we have the result. (2) follows from (1). (3) By (1), diagrams for cyclic modules are connected. If W_V is not connected, two submodules generated by primitive vectors in two disjoint parts are disjoint and V is their disjoint sum. ■

In $\overline{V}(\Lambda, n)$, all $v_{\Lambda^-}^{(i)}$ have weight $\Lambda^- < \Lambda$, thus, Lemma 3.1 does not hold for generalized weight modules; and we have $v_{\Lambda^-}^{(i)} \in G_{-1}U(G^-)U(H)v_{\Lambda}^{(n)}$, but $v_{\Lambda^-}^{(i)} \notin G_{-1}U(G^-)v_{\Lambda}^{(n)}$ if $i < n$, thus by (2.1), $v_{\Lambda^-}^{(i)}$ is not linked by the generator $v_{\Lambda}^{(n)}$ if $i < n$. However, from the structure of weight diagrams of cyclic modules in Lemma 3.6, we see that Corollary 3.2 still holds for generalized weight diagrams. Now similar (but not exactly) to Ref. 4, we define

Definition 4.7. (1) A *chain* is a triple (C, Λ, x) , where Λ is a singly atypical weight, C is a finite set: $C = \{v_1, \dots, v_n\}$, $x \in \mathbb{C}$, such that v_i has a weight $\lambda_i = \Lambda^{(k)}$ for some k and there are links with arrows between elements in C , such that

- (i) each element v is linked by at most 2 elements, and in this case, it can only be one of (a) $\leftarrow v \rightarrow$, (b) $\rightarrow v \leftarrow$, (c) $v \leftarrow$, (d) $\rightarrow v$ (accordingly, v is called a *top*, *bottom*, *left*, *right*, *point*);
- (ii) If v_i is linked to v_j with an arrow pointed to it, then v_i is not derived from v_j (we say v is *derived* from u if there exist $v = u_0, u_1, \dots, u_k = u$ such that u_{i+1} is linked to u_i with an arrow pointed to it for $i = 0, \dots, k-1$);
- (iii) C is connected;
- (iv) if v_i is linked to v_j from the left, then $\lambda_i = \lambda_j^+$;
- (v) if v_i is a *leftmost* element, i.e., there is no elements linked to it from left side, then $\lambda_i = \Lambda$;
- (vi) $x = 0 \Leftrightarrow$ there are 2 endpoints.

(2) A *subchain* of (C, Λ, x) is a subset of C together with the weight, the original relationship between the elements.

- (3) An *isomorphism* between two chains (C, Λ, x) and (C', Λ', x') is a bijection $C \rightarrow C'$ which preserves weight, linking relationship and $x = x'$.
- (4) An *anti-isomorphism* between (C, Λ, x) and (C', Λ', x') is a bijection $\phi : C \rightarrow C'$ such that (i) $u \leftarrow v \Leftrightarrow \phi(v) \rightarrow \phi(u)$, and $u \rightarrow v \Leftrightarrow \phi(v) \leftarrow \phi(u)$; (ii) if a *rightmost* element of C' has the weight λ , then $\lambda_{low} = -\Lambda$; (iii) $x = x'$. ■

Remark 4.8. (1) For a chain (C, Λ, x) , we can break it, at top and bottom points, into pieces of subchains according to the rule: if $\leftarrow v_\lambda \rightarrow$ (or $\rightarrow u_\lambda \leftarrow$), then we break at the point v_λ (or u_λ) into $\leftarrow v'_\lambda, v''_\lambda \rightarrow$ (or $\rightarrow u'_\lambda, u''_\lambda \leftarrow$, resp.). Then each piece is corresponding to some $\overline{V}_i(\Lambda^{(j)}, k)$. If we only break it at the bottom points, then each piece corresponds to a cyclic module. This gives us a better understanding of what a chain is.

- (2) If a chain is a line, then conditions (ii)&(iv) are unnecessary as (ii) can not happen and we have a unique way to associate with each v_i a weight λ_i such that (iv) is satisfied.
- (3) Examples of non-chains: (a) a diagram such that $v_1 \rightrightarrows v_2$ and $v_3 \leftrightsquigarrow v_4$ (it violates (ii)); (b) $v_1 \rightrightarrows \leftrightsquigarrow v_5$ (it violates (iv)). ■

We see that the weight diagrams of all indecomposable modules introduced up to now are chains. Now we can prove the main result of this section.

Theorem 4.9. (1) For any chain (C, Λ, x) , there is a unique indecomposable generalized weight G -module $X(C, \Lambda, x)$ corresponding to it.

(2) If V is an indecomposable generalized weight module with a primitive weight Λ of typical or singly atypical type, then (i) $V = \overline{V}(\Lambda, n)$ if Λ is typical, or (ii) there exists a unique chain (C, Λ, x) , up to an (anti-) isomorphism, such that $V = X(C, \Lambda, x)$ if Λ is singly atypical.

Proof. (1) Break the chain as in Remark 4.8, let V' be the direct sum of all $\overline{V}_i(\Lambda^{(j)}, k)$ obtained. Then for each pair of u'_λ, u''_λ , we can “merge” them into one u_λ (this is a bottom point) by taking quotient $V'/U(G)(u'_\lambda - u''_\lambda)$ and let V'' be this quotient module; and for each pair of v'_λ, v''_λ , we “join” them into one v_λ (this is a top point) by letting $v_\lambda = v'_\lambda + av''_\lambda$ (where $a = x$ if $x \neq 0$ and v_λ is the last vector to be joined in order to form the circle; or otherwise, $a = 1$). Now let $X(C, \Lambda, x)$ be the submodule of V'' generated by all “joined” vectors v_λ (all top points). Then we see $X(C, \Lambda, x)$ is corresponding to (C, Λ, x) . By the statements following (4.11), we see that it is uniquely determined by (C, Λ, x) . To see it is indecomposable, suppose it is a direct sum of submodules $V_1 \oplus V_2$. If v_λ plus a linear combination of other vectors in C with weight λ belongs to V_1 , then by Lemma 4.6(1.ii), all vectors derived from v_λ is in V_1 ; as C is connected, all vectors in C must also be in V_1 ; thus $V_2 = 0$.

(2)(i) Let V' be the submodule generated by $\{v_\lambda \in C \mid \lambda \text{ singly atypical (if any)}\}$, and let $V^{(\lambda)}$ be the submodule generated by $\{v \in C \mid v \text{ has weight } \lambda\}$ if λ is typical. Then by Lemma 4.6(1.i),

$V = \oplus_{\{\lambda \text{ typical}\}} V^{(\lambda)} \oplus V'$ is a direct sum, thus $V = V^{(\Lambda)}$. Now by Lemmas 4.6(1.i) & 4.5(1.i), we have $V = \overline{V}(\Lambda, n)$.

(ii) (cf. proof of Theorem 3.6.(1)) We want to prove *Statement (A)*: For any generalized weight module V , we can choose W_V such that it is a union of pieces of disconnected subdiagrams, each subdiagram is a chain. Then by Lemma 4.6(3), we have the result. Now let V has a composition series as in (4.2). If $n \leq 2$, (A) follows immediately from Lemma 4.5. Assume now $n \geq 3$. By induction, suppose $W_{V^{(n-1)}}$ satisfies (A). Let v be a primitive vector corresponding to $V/V^{(n-1)}$. Then v is not derived by any primitive vector. To understand it better, we prove (A) case by case. Using Corollary 3.2, we consider all possible cases below. *Case (a)*: If v is not linked to $W_{V^{(n-1)}}$, then v itself is a piece of chain. Thus we have (A). *Case (b)*: If v is linked to $W_{V^{(n-1)}}$ only on the right side, then there is a unique u in a piece of $W_{V^{(n-1)}}$ such that $u \leftarrow v$. Thus, there does not exist u' such that $u' \leftarrow u$. *(b.1)*: If there is $v' \in W_{V^{(n-1)}}$ such that $u \leftarrow v'$, then we can replace v by $v - v'$ so that v is now not linked to $W_{V^{(n-1)}}$. This becomes (a). *(b.2)*: It remains that u is either a left top endpoint: $u \rightarrow w$, or a right bottom endpoint: $w \rightarrow u$. Then we see that v can be added to that piece so that we have $u \xrightarrow{\leftarrow v} w$ or $w \rightarrow u \leftarrow v$, and it is still a chain. Thus we have (A). *Case (c)*: Similar to (b), if v is linked to $W_{V^{(n-1)}}$ only on the left side, we have (A). *Case (d)*: If v is linked to $W_{V^{(n-1)}}$ on both sides, i.e., there are unique $u, w \in W_{V^{(n-1)}}$ such that we have *Diagram (D.1)*: $u \leftarrow v \rightarrow w$. *(d.1)*: If u, w belong to 2 different pieces of chains, then as in (a) and (b), by a suitable choice of v and rescaling primitive vectors if necessary, either v can be added to one piece, or we can link 2 pieces through v into one piece of chain. Thus we have (A). *(d.2)*: Finally suppose u and w are in the same piece of chain. In this case, suppose v has weight λ , and $u = g_\lambda^+ v$, $g_\lambda^- v = xw$ for some $x \neq 0$. *(d.2.1)*: If there is $v' \in W_{V^{(n-1)}}$ such that we have *Diagram (D.2)*: $u \leftarrow v' \rightarrow w$. *(d.2.1.i)*: If $x = 1$. Replace v by $v - v'$, then v is not linked to any one. This becomes (a) and we have (A). *(d.2.1.ii)*: If $x \neq 1$, by replacing v by $v - v'$ and replacing v' by $xv - v'$, when we add v into that piece, from (D.1) and (D.2), we see that (D.2) is broken into: $u \leftarrow v'$, $v \rightarrow w$ (cf. the statement before Lemma 4.6). Therefore it is still corresponding to a piece of chain (or 2 pieces of chains if that piece becomes disconnected). Thus we have (A). *(d.2.2)*: If there exists $v' \in W_{V^{(n-1)}}$ such that $v' \rightarrow w$, but not $u \leftarrow v'$ or there exists $v'' \in W_{V^{(n-1)}}$ such that $u \leftarrow v''$ but not $v'' \rightarrow w$, or if both, by replacing v by $v - v'$, or $v - v''$, or $v - v' - v''$, we see that (D.1) becomes $u \leftarrow v$ or $v \rightarrow w$, or v . This becomes (b) or (c) or (a). *(d.2.3)*: It remains that u is either a right bottom endpoint $t \rightarrow u$ or a left top endpoint $u \rightarrow t$ and w is either a left bottom endpoint $w \leftarrow x$ or a right top endpoint $x \leftarrow w$. From (D.1), we see that v can be added to it so that we have $t \rightarrow u \leftarrow v \rightarrow w \leftarrow x$ or $t \rightarrow u \xleftarrow{\leftarrow v} x \leftarrow w$, or $u \xrightarrow{\leftarrow v} t \leftarrow w \leftarrow x$ or $u \xrightarrow{\leftarrow v} t \xleftarrow{\leftarrow w} x$, and it is still a chain. Thus we have (A). This completes the proof of *Statement (A)*. ■

Remark 4.10. By applying the above results to $sl(m/1)$, as all primitive weights of $sl(m/1)$ are typical

or singly atypical, we have efficiently classified all finite dimensional (weight or generalized weight) $sl(m/1)$ -modules. ■

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